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On Relations Between Constants in Homogeneous Turbulence Models and Heisenberg's Spectral Model

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Abstract

Recently, Muschinski and Roth (1993) in this journal deduced the coefficient $c_\mu = \nu_t \varepsilon / E^2$ as required for energy-dissipation closure models using Heisenberg's spectral eddy-viscosity model. It is shown that this coefficient can be determined by using the inertial subrange theory of Kolmogorov for local equilibrium without referring to the viscous subrange, where Heisenberg's model has only limited validity. Alternative models for the viscous range are also discussed and compared to available measurements of energy and dissipation spectra and of the velocity derivative skewness. None of the common models represents the observed features satisfactorily.

Zusammenfassung

Über Beziehungen zwischen Konstanten in Turbulenz-Modellen für homogene Turbulenz und Heisenbergs Spektralmodell

Muschinski and Roth (1993) haben kürzlich in dieser Zeitschrift den Koeffizienten $c_\mu = \nu_t \varepsilon / E^2$ für Energie-Dissipations-Modelle abgeleitet und dazu Heisenbergs spektrales Modell für die turbulente Diffusivität benutzt. Es wird gezeigt, daß sich dieser Koeffizient auf der Basis der Theorie von Kolmogorov für das Spektrum in Trägheitsbereich im lokalen Gleichgewicht ableiten läßt, ohne Bezug auf den viskosen Bereich, wo Heisenbergs Modell nur begrenzt gültig ist. Es werden alternative Modelle für den viskosen Bereich diskutiert und mit vorhandenen Meßdaten für Energie- und Dissipations-Spektren und für die Schiefe von Geschwindigkeitsableitungen verglichen, aber keines der üblichen Modelle beschreibt die beobachteten Eigenschaften zufriedenstellend.

1 Introduction

In a recent paper, Muschinski and Roth (1993) (abbreviated as MR) derived relationships between various constants in homogeneous turbulence closure models based on Heisenberg's eddy-viscosity model (Heisenberg, 1948). Such relations are important to be known for practical applications of turbulence models and for understanding of intrinsic relationships between various theories. As discussed by MR, Heisenberg's heuristic theory has often been criticized (see also Hinze, 1959, p. 192–193 and 195–196). Nevertheless, MR use this theory because it gives an explicit equation for the turbulent diffusivity as a function of the energy spectrum.

This note discusses this theory in two respects. First, we point out that the relationships given by MR can be derived without the need of referring to Heisenberg's theory. In fact, Heisenberg's diffusivity model is consistent with and can be deduced from theories based on the inertial subrange concept of Kolmogorov (1941) and the condition of local equilibrium between production and dissipation of small-scale turbulent motion energy. We will see that these relations are closely related to subgrid-scale models as used for large-eddy simulations (Lilly, 1967).

Second, we will discuss alternative spectral models like those proposed by Kraichnan (1959), Pao (1965), and Orszag et al. (1993). Contrary to what was to be expected, in view of the cited criticism, it is found that Heisenberg's spectrum fits the spectrum in the range of dissipating scales quite well. It does poorly with respect to fourth order moments of the spectrum, but the other models are not better.

2 Derivation of Model Coefficients

As one of their main results, MR deduce the coefficient

$$c_\mu = \frac{\nu_t \varepsilon}{E^2} = \left(\frac{2}{3\alpha}\right)^3 \tag{1}$$

as a function of the Kolmogorov coefficient $\alpha \cong 1.5 \pm 0.1$ (Champagne, 1978; Saddoughi and Veeravalli, 1994, abbreviated as SV). Here, ν_t is the turbulent viscosity, E is the kinetic energy of the turbulent motions, and ε is the viscous dissipation rate of this energy. As in Heisenberg (1948), MR assume that the spectrum of kinetic energy F (k) is zero or very small at low wavenumbers k below the wavenumber k_0 of most energetic motions, and follows Heisenberg's spectrum (see Batchelor, 1959, eq. 6.6.15) otherwise. In the inertial subrange, i.e. for $k_0 << k << k_d$, this spectrum equals Kolmogorov's law

$$F(k) = \alpha \varepsilon^{2/3} k^{-5/3}. \tag{2}$$

In the viscous subrange, for $k > k_d$, the energy decays more strongly. The limiting wavenumber k_d is proportional to the inverse of Kolmogorov's lengthscale $\eta = (\nu^3/\varepsilon)^{1/4}$ as a function of the molecular viscosity ν. MR evaluate Eq. (1) on the basis of Heisenberg's eddy-viscosity model, which was derived from dimensional arguments, and which states

$$\frac{\partial \nu_t}{\partial k} = -c_H \left(\frac{F(k)}{k^3}\right)^{1/2}. \tag{3}$$

(The minus sign has to be inserted in MR.) Heisenberg (1948) gave this relation in its integral form, and required that his coefficient c_H is a universal constant for all $k > k_0$. As will be shown below, c_μ can be derived without knowing the energy spectrum in the viscous range. We remark that the result of MR for c_μ is also independent on the viscous parts of the spectrum, in spite of the fact that they include the molecular viscosity in its derivation.

We base our derivation on the concept of large-eddy motions at $k < k'$, and "subgrid scale" motions at $k \geq k'$, with $k_0 << k' << k_d$, as introduced by Lilly (1967). See also Deardorff (1971), Moeng and Wyngaard (1989), Schmidt and Schumann (1989), Schumann (1975, 1991), and the recent remarks on the history of this approach by Smagorinsky (1993). Heisenberg's approach is similar in this splitting of the wavenumber range. In finite difference approximations, the "cutoff wavenumber" k' is usually related to a grid scale Δ by

$$\Delta = \pi/k'. \tag{4}$$

For convenience, we refer to the results as given in appendix B of Schmidt and Schumann (1989):

$$\varepsilon = c_\varepsilon \frac{e^{3/2}}{\Delta}, \quad \nu_t = c_\nu \Delta e^{1/2}, \tag{5}$$

with

$$c_\varepsilon = \left(\frac{2}{3\alpha}\right)^{3/2} \pi, \quad c_\nu = \left(\frac{2}{3\alpha}\right)^{3/2} \frac{1}{\pi}. \tag{6}$$

Here e is the kinetic energy of subgrid-scale motions, denoted by E(k′) in MR, and ν_t is the turbulent diffusivity by which subgrid-scale motions transport large-eddy momentum. Both e and ν_t are functions of Δ or k'. The turbulent diffusivity causes transfer of kinetic energy from the large eddy velocity field $\bar{u}_i(t, x_j)$ to subgrid scale motions at a rate

$$S = \frac{1}{2} \nu_t (\partial \bar{u}_i/\partial x_j + \partial \bar{u}_j/\partial x_i)^2. \tag{7}$$

The given result for the coefficient c_ε follows simply from

$$e = \int_{k'}^{\infty} F(k)\, dk = \gamma \varepsilon^{2/3} k'^{-2/3} =$$

$$= \frac{3}{2} F(k') k' = \left(\frac{\varepsilon \Delta}{c_\varepsilon}\right)^{2/3}, \tag{8}$$

with $\gamma = 3\alpha/2$ as in MR. The result for the coefficient c_ν can be verified assuming that the production S of subgrid-scale energy is balanced by the rate of dissipation ε in the ensemble mean, with negligible molecular effects in the inertial subrange,

$$S = 2\nu_t \int_0^{k'} k^2 F(k)\, dk = \varepsilon. \tag{9}$$

This relation follows from Eq. (7) for isotropic turbulence. The integral is evaluated assuming that F(k) equals the inertial subrange spectrum for all wavenumbers down to k = 0, but variations at small wavenumbers are of minor importance when $(k_0/k')^{4/3} << 1$, because of the multiplier k^2 in the integral.

From Eqs. (5) and (6), it is straightforward to obtain

$$c_\mu = \frac{\nu_t \varepsilon}{e^2} = c_\varepsilon c_\nu = \left(\frac{2}{3\alpha}\right)^3. \tag{10}$$

The factor π, resulting from Eq. (4), cancels in this product. This reveals the fact that the precise relation between Δ and k' is irrelevant for c_μ. Since c_μ is independent of k', it should formally also apply to the total turbulent kinetic energy $E = e(k_0)$. Hence, we find the same result as MR, but without using Eq. (3). However, since the basic assumptions of local isotropy and inertial range spectrum are not valid at small wavenumbers, the precise value of c_μ for the full turbulence regime will differ slightly from this inertial subrange result.

The application of this kind of theory to the Prandtl layer, as in MR, with evaluation of the von Karman constant κ as a function of α requires a relationship between vertical height z above surface and k'. For this purpose, MR introduce $q = zk'$ as a new empirical coefficient, and determine $q \cong \pi/2$ such that the resultant relation between α and κ is consistent with observations.

It may be noted that the von Karman constant is related also to the subgrid-scale coefficient of Smagorinsky, $c_S = [2/(3\alpha)]^{3/4}/\pi$, by $\kappa z \cong c_S \Delta$ (Deardorff, 1971; Smagorinsky, 1993). From this, the same relation between κ and α, namely $\kappa = 2c_S$ results as in MR if $z = \Delta/2$, which is a standard assumption near walls in large-eddy simulations (Schumann, 1975).

3 On Heisenberg's Relationship

Heisenberg's relationship and the coefficient c_H can be deduced as a consequence of Eq. (5) for ν_t:

$$\nu_t = c_\nu \Delta e^{1/2} = \beta \varepsilon^{1/3} k'^{-4/3}, \tag{11}$$

with $\beta = 2/(3\alpha)$ as in MR. By differentiation one obtains

$$\frac{\partial \nu_t}{\partial k'} = -\frac{4}{3}\beta\varepsilon^{1/3}k'^{-7/3} = -c_H\left(\frac{F(k')}{k'^3}\right)^{1/2}, \tag{12}$$

i.e. Eq. (3), with Heisenberg's coefficient $c_H = (8/9)\,\alpha^{-3/2}$. The same relation between c_H and α follows by comparing Heisenberg's spectrum with the Kolmogorov law (Batchelor, 1959; Rotta, 1972, p. 98).

One should note, that the concept of constant eddy viscosity of all motion scales is valid only for clearly separated resolved and subgrid scales, i.e. for $k << k'$. Otherwise, the effective viscosity depends on both k and k'. As shown by Chasnov (1991), shortly below the cutoff wavenumber k' the turbulent viscosity increases strongly with k and energy gets backscattered from large to small wavenumbers. These effects are neither covered by Heisenberg's model nor by the above inertial range theory. However, Zhou (1993) shows that nonlocal interactions are more important in the far-dissipation range than in the inertial range.

As explained in MR, the Heisenberg spectrum follows from Eq. (3) and the condition of local equilibrium between production and dissipation,

$$F(k) = \alpha\varepsilon^{2/3}k^{-5/3}\left(1 + \frac{27}{8}\alpha^3\eta^4k^4\right)^{-4/3}. \tag{13}$$

For large wavenumbers, $k >> k_d$, this spectrum approaches the power law $F(k) \sim k^{-7}$. Heisenberg (1948) mentioned that this spectrum may not be valid very far in the viscous regime, see also Rotta (1972, p. 99–100). In fact, such a power law is physically unrealistic if extended to very large wavenumbers for several reasons (Hinze, 1959). One obvious objection (Stewart and Townsend, 1951) is that moments $\int_0^\infty k^n F(k)\,dk$ of the spectrum become infinite for $n \geq 6$; implying the non-existence of mean-square third- and higher-order velocity derivatives (Kraichnan, 1959). There is no reason to assume that high-order derivatives are non-decaying far in the viscous range (and long before the scale of molecular motions). High-order derivatives are difficult to measure, but no data exist which would indicate that they are becoming infinite (Stewart and Townsend, 1951; Champagne, 1978; Sreenivasan, 1985). Direct numerical simulations of turbulence exhibit finite moments even though the small scale motions are highly intermittent (Chen et al., 1993). Anselmet et al. (1984) measured velocity correlations up to eighteenth order at high Reynolds numbers $R_\lambda \leq 852$, showing finite values. ($R_\lambda = u'\lambda/\nu$ with Taylor's miroscale $\lambda = (15\nu u'^2/\varepsilon)^{1/2}$ and mean turbulent velocity fluctuations $u' = (2E/3)^{1/2}$.)

One possible reason for the unsatisfactory behaviour of Heisenberg's spectrum in the viscous subrange, which was also seen by Heisenberg (1948), is due to the fact that Eq. (3) was derived on the assumption that ν_t depends solely on $F(k')$ and k'. If that would be true, dimensional arguments lead uniquely to the result given in Eq. (3). However, in the viscous subrange, the turbulent motions depend also on the molecular viscosity, so that there ν_t is a function of ν in addition to $F(k')$, and k'. Hence, dimensional arguments can no longer be used to derive such a unique relationship. One has to conclude that Heisenberg's theory has no solid base for the viscous subrange, its predictions have to be taken with care, and are certainly wrong for high-

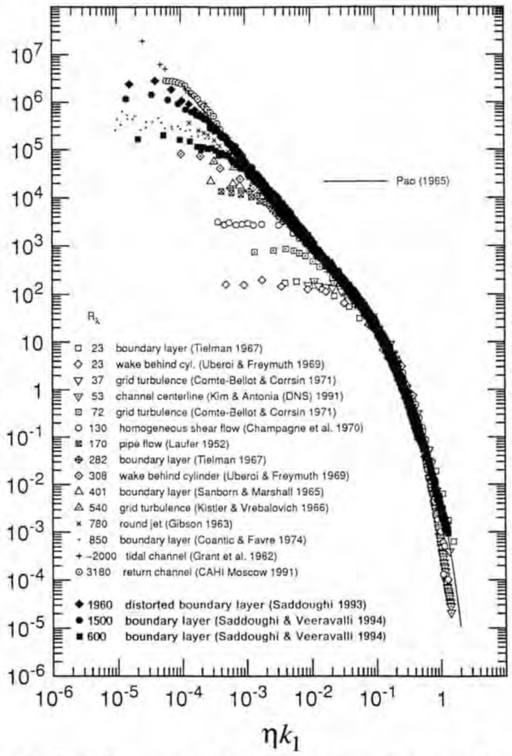

Figure 1 One-dimensional longitudinal energy spectra of turbulent flows at various Reynolds numbers R_λ in Kolmogorov's scales compared to Pao's (1965) spectral model, from Saddoughi and Veeravalli (1994) with additions from Saddoughi (1993). The graph has been kindly provided by S. G. Saddoughi.

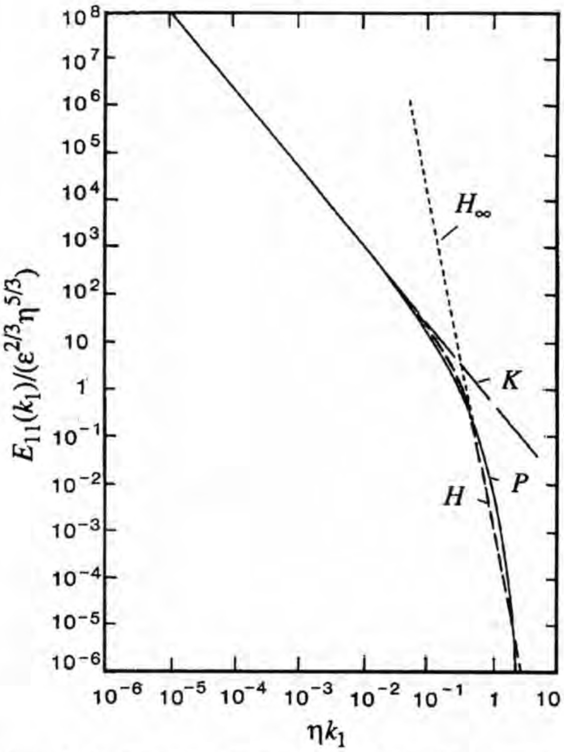

Figure 2 One-dimensional spectra as in Figure 1, showing the inertial subrange spectrum K: $(18/55)\,\alpha(\eta k_1)^{-5/3}$, together with model spectra P: Pao (1965), H: Heisenberg (1948), and the viscous asymptote of Heisenberg's spectrum H_∞: $\alpha^{-3}(32/5103)\,(\eta k_1)^{-7}$, using $\alpha = 1.5$.

order moments of the spectrum. In the inertial subrange, its results, as we have shown, are equivalent to other inertial range theories.

4 Alternative Spectral Models for the Viscous Subrange

As shown above, the viscous subrange is of little relevance for practical turbulence models of high Reynolds number flows. However, the spectrum in the viscous subrange is of theoretical interest. It is also of practical interest, e.g., for computation of turbulent motions of particles or cloud droplets which are of a size comparable to the Kolmogorov scale η. Because of the limitations of Heisenberg's theory it is of interest to see whether alternative models are more convincing.

By dimensional analysis (with similar limitations as in Heisenberg's approach, see Tennekes and Lumley, 1972, p. 269), Pao (1965) deduced an alternative transfer model which results in a spectrum

$$F(k) = \alpha\,\varepsilon^{2/3} k^{-5/3} \exp\left[-(3/2)\,\alpha\,(\eta k)^{4/3}\right] \qquad (14)$$

for the transition from the inertial subrange to the viscous subrange. Near $\eta k \cong 1$, his spectrum predicts about three times larger values of F than Heisenberg's spectrum, but Pao's spectrum decays exponentially and becomes much smaller for large wavenumbers. As a consequence, any higher order even moment of the velocities with such spectrum stays finite, which is a definite advantage of Pao's model. However, it remains to show which model compares better with data for the lower order moments.

Figure 1, taken from SV, depicts a large set of data for the one-dimensional spectrum E_{11} as a function of the down-stream wavenumber k_1. For the definitions see Hinze (1959, eq. 3-72). The figure shows nicely the scaling of spectra in many different flow

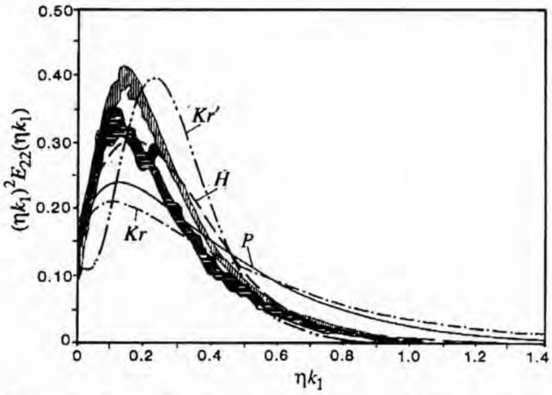

Figure 3 Normalized cross-component velocity spectrum E_{22} (or E_{33}) versus downstream wavenumber k_1 for the viscous subrange. Data from Champagne (1978) for $R_\lambda = 182$ are within the vertically hatched region, data from SV for $R_\lambda = 600$ in the horizontally hatched region. The curves show the model spectra of P: Pao (1965), H: Heisenberg (1948), Kr: Kraichnan (1959) with $\alpha' = 0$, and $\mu = 2.094$, and Kr': Eq. (15) with $\alpha' = 47000$, and $\mu = 16.2$, all for $\alpha = 1.5$.

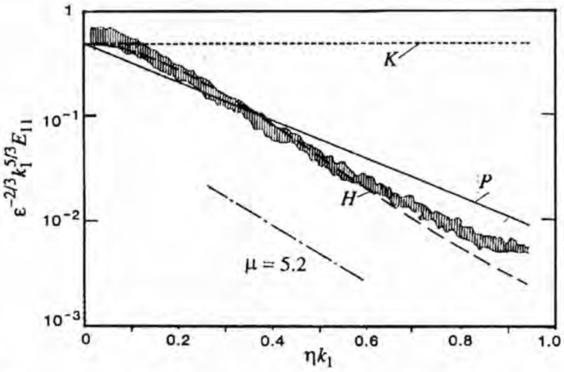

Figure 4 Log-linear plot of the normalized spectrum $E_{11}(k_1)$ for boundary layer turbulence with $R_\lambda = 600$. Data from Saddoughi and Veeravalli (1994). The horizontal line K depicts the inertial range behaviour, the dash-dotted line identifies an exponential spectrum with $\mu = 5.2$. The curves show the model spectra of P: Pao (1965) and H: Heisenberg (1948) for $\alpha = 1.5$.

configurations at various Reynolds numbers with a clear inertial subrange as postulated by Kolmogorov (1941), at least for $R_\lambda \geq 100$. It also suggests a good agreement of the data with Pao's model. However, similar agreement results if we compare the same data with Heisenberg's spectrum in such logarithmic scales, see Figure 2. For a conclusive test of the spectral decay at large wavenumbers, data would be required for $\eta k_1 > 3$.

For small Reynolds number turbulence, Kraichnan (1959) deduced from a statistical theory that the spectrum varies as $k^3 \exp(-\mu\eta k)$. Orszag et al. (1993) generalized this finding and postulated that the "Kraichnan-spectrum" behaves as

$$F(k) = \alpha \varepsilon^{2/3} k^{-5/3} [1 + \alpha' (\eta k)^{14/3}] \exp(-\mu\eta k), \quad (15)$$

with new coefficients $\alpha' = O(10^{-1})$, and $\mu = O(10)$. The exponential decay at large wavenumbers is supported by several experimental and numerical turbulence data but with different values of μ, see Sreenivasan (1985), Chen et al. (1993), and Zhou (1993). SV found that $\mu = 5.2$ fits their data in the range $0.5 \leq \eta k_1 \leq 1$ (see below, Figure 4). However, when Kraichnan's spectrum is applied to the whole inertial and viscous ranges, the value of μ cannot be taken arbitrarily. The integral identity $\varepsilon = 2\nu \int_0^\infty k^2 F(k)\, dk$ for isotropic turbulence requires that

$$\alpha' = \frac{\mu^6}{240\alpha} - \frac{\Gamma(4/3)\,\mu^{14/3}}{120}, \quad (16)$$

with the Gamma-function value $\Gamma(4/3) = 0.89298$. For $\alpha' = 0$, this implies $\mu = [2\alpha\Gamma(4/3)]^{3/4} \cong 2.094$ (for $\alpha = 1.5$).

As shown by SV and others, measured spectra plotted in the form $\varepsilon^{-2/3} k_1^{5/3} E_{11}(k)$ versus ηk_1, exhibit a local maximum, a "bump", in between $0.02 \leq \eta k_1 \leq 0.15$. exceeding the expected value $18\alpha/55$ of the inertial subrange by about 20 %. As explained by Mestayer et al. (1984), the local maximum indicates a retardation of energy transfer when passing from the inertial to the viscous range. This bump may also explain why the effective value of α is found to be rather large in low Reynolds number turbulence (Chasnov, 1991). The spectral bump gives some support to using Eq. (15) with $\alpha' > 0$.

Figure 3 compares various theoretical spectra with data of Champagne (1978) and SV for normalized cross-stream spectra E_{22} (Hinze, 1959, eq. 3-73) in linear scales. The integral of this normalized energy dissipation spectrum amounts to 2/15. We see that Champagne's data suggest a larger value of the Kolmogorov coefficient than SV's data, possibly due to differences in Reynolds number. The rather large values near $\eta k_1 \cong 0.1$ are related to the mentioned bump. We find that neither Pao's nor Kraichnan's spectrum (for $\alpha' = 0$) fits the data well. For $\alpha' > 0$, the E_{11} and E_{22} spectra resulting from $F(k)$ according to Eq. (15) show values which are considerably smaller than the Kolmogorov spectrum at low wavenumbers. The spectrum does show a "bump" but at too high wavenumbers. Incidentally, Heisenberg's spectrum comes closest to the data in this range, but does not represent the bump near $\eta k_1 \cong 0.1$. This is also corroborated by Figure 4, exhibiting a log-linear plot of data from SV which was used to support Kraichnan's spectrum with $\mu = -5.2$. Obviously, also Heisenberg's spectrum is close to the purely exponential decay in the limits $0.3 \leq \eta k_1 \leq 1$. Hence, at least with respect to dissipation spectra, Heisenberg's spectrum provides a rather well fitting (but theoretically unjustified) interpolation formula.

As shown by Lee (1950), Batchelor (1959, p. 168) and Tavoularis et al. (1978), the viscous part of the energy spectrum is directly related to the skewness s of velocity derivatives,

$$s \equiv - \frac{\overline{(\partial u/\partial x)^3}}{\left[\overline{(\partial u/\partial x)^2}\right]^{3/2}} =$$

$$= \frac{12}{7} (15)^{1/2} \nu^{5/2} \varepsilon^{-3/2} \int_0^\infty k^4 F(k)\, dk\,. \qquad (17)$$

This relation results from the steady state budget of enstrophy and shows that the skewness is a measure of energy transfer from small to large wavenumbers (Batchelor, 1959). A suitable spectral model should, therefore, give the correct value for this important parameter.

The various spectra $F(k)$ result in $s \cong 1.35\alpha^{-3/2}$, $s = (3/7)\,(10\pi)^{1/2}\alpha^{-3/2}$, and $s = (80/6)^{1/2}\{\Gamma(4/3)\,\alpha\}^{-3/2}$ for the models of Heisenberg, Pao, and Kraichnan, respectively (using $\alpha' = 0$, and μ according to Eq. (16)). For $\alpha = 1.5$, this implies $s \cong 0.735$, 1.31, 2.36, for the respective models. Measurements in laboratory turbulent flows at R_λ of the order 10 to 100, give $s \cong 0.45 \pm 0.05$ (Tavoularis et al., 1978). Recently, S. G. Saddoughi (personal communication, 1994) measured $s \cong 0.56$ for $R_\lambda = 500$ and 600. The measured values are much smaller than the model predictions. For $\alpha' > 0$, Eq. (15) implies a skewness

$$s = (4/3)\,(15)^{1/2}[27 - 50\Gamma(4/3)\,\alpha\mu^{-4/3}]\,\mu^{-2}.$$

If μ is fitted to give the expected skewness value of $s \cong 0.5$, then one obtains $\mu = 16.2$ and $\alpha' \cong 47000$. These parameter values are much larger than expected by Orszag et al. (1993), and the corresponding spectrum, as shown in Figure 3, does not fit the observations.

All the model spectra give skewness values which are independent of the Reynolds number, at least for $(k_0\eta)^{10/3} \sim R_\lambda^{-5} << 1$. However, data from atmospheric boundary layers indicate that s increases slowly with R_λ for $R_\lambda > 1000$. Up to now, the maximum measured value of skewness amounts to about 0.9 for $R_\lambda \cong 13000$ (Champagne, 1978). Such variations are not explainable with the given spectral models. The large skewness values at high Reynolds numbers have been explained with the spatially intermittent distribution of dissipation (Wyngaard and Tennekes, 1970). With respect to second-order velocity moments in the inertial range and for the energy spectrum, the intermittency effects are small (Anselmet et al., 1984) and non-detectable in plots like Figure 1.

Hence, none of the simple models explains the complete spectrum from the inertial subrange into the far-dissipation range. In particular, the models do not account for the bump in $k^{5/3}F(k)$ near $\eta k_1 \cong 0.1$ and for intermittency effects on the skewness at large Reynolds numbers. For practical interpolation purposes, up to about $\eta k \leq 2$, Heisenberg's spectrum is at least as suited as other theories. However, Heisenberg's theory should not be used in the viscous subrange for theoretical predictions of new relationships. It is remarkable that more than 45 years after the publication of Heisenberg's paper a satisfactory simple spectral model for the turbulence at dissipating scales is still missing.

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